Homework 4
Geometry

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Definition 0.1. *Let X, Y be topological spaces and let $f : X \to Y$. We say that f is* ***locally constant*** *if for every $x \in X$, there exists a neighborhood U_x such that $f|_{U_x}$ is a constant function.*

Lemma 0.2 (for Exercise 3-1)**.** *Let $f : X \to Y$ be a locally constant function and X a connected space. Then f is constant on X.*

Proof. For each $x \in X$, let U_x be an open neighborhood of x such that f is constant on U_x. Pick some $y \in f(X)$, and set

$$U = \bigcup \{U_x : f(x) = y\}$$
$$V = \bigcup \{U_x : f(x) \neq y\}$$

Then both U, V are unions of open sets, so they are open. We also know that $U \cup V = X$ and $U \cap V = \emptyset$. Since $y \in f(X)$, U must be non-empty. Thus since X is connected, V must be empty (since we cannot write X as a disjoint union of non-empty open sets). Thus $U = X$, so f is constant on X. $\square$

Corollary 0.3 (for Exercise 3-1)**.** *Let $f : X \to Y$ be a locally constant function. Then f is constant on each connected component of X.*

Proof. Let A be a connected component of X. Then $f|_A : A \to Y$ is locally constant and A is connected, so $f|_A$ is constant by the above lemma. $\square$

Proposition 0.4 (Exercise 3-1)**.** *Let M, N be smooth manifolds, and let $F : M \to N$ be smooth. Then F is constant on each component of M if and only if $dF_p : T_pM \to T_{F(p)}N$ is the zero map.*

Proof. First suppose that F is constant on each component of M. Let $p \in M, v \in T_pM, f \in C^\infty(M)$. Let U_p be the component of M containing p. Then $f \circ F$ and $f \circ F|_{U_p}$ agree on U_p, so $f \circ F|_{U_p}$ is constant. Thus

$$dF_p(v)(f) = v(f \circ F) = v(f \circ F|_{U_p}) = 0$$

using Proposition 3.8 and Lemma 3.4a. Thus $dF_p(v)$ is the zero function for each $v \in T_pM$, so dF_p is the zero map. Since p is arbitrary, this holds on each component of M.

Now suppose that dF_p is the zero map. Let $p \in M$, and let (U, ϕ) be a chart for M with $p \in U$ and let (V, ψ) be a chart for N with $F(p) \subset V$. Let $\widehat{F} = \psi \circ F \circ \phi^{-1}$ be the coordinate representation of F and let $\widehat{p} = \phi(p)$. Since $dF_p(v) = 0$ for all $v \in T_pM$, we have

$$0 = dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \frac{\partial \widehat{F}}{\partial x^i}(\widehat{p}) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}$$

and since

$$\left\{ \left. \frac{\partial}{\partial y^j} \right|_{F(p)} \right\}$$

is a basis for $T_{F(p)}N$, it must be that each coefficient $\frac{\partial \widehat{F}}{\partial x^i}(\widehat{p})$ is zero. Since p is arbitrary in U, this holds for all $p \in U$, so for all $\widehat{p} \in \phi(U)$. So our coordinate representation $\widehat{F}$ is a function between Euclidean spaces with all partial derivatives vanishing on $\phi(U)$, so we know from calculus that $\widehat{F}$ is constant on $\phi(U)$. Thus F is constant on U. Since $p \in M$ is arbitrary, F is locally constant on M. Thus F is constant on each component of M by the previous corollary. □

Lemma 0.5 (for Exercise 3-2)**.** *A function has a right inverse if and only if it is surjective. (Note: This depends on the Axiom of Choice.)*

Proposition 0.6 (Exercise 3-2)**.** *Let $M_1, \ldots M_k$ be smooth manifolds. For $j = 1, \ldots k$ let $\pi_j : M_1 \times \ldots \times M_k$ be the projection $(p_1, \ldots p_k) \mapsto p_j$. Then define*

$$\begin{aligned} &\alpha : T_p(M_1 \times \ldots \times M_k) \to T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k \\ &\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)) \end{aligned}$$

The map α is an isomorphism. Furthermore, if one of the M_i is a smooth manifold with boundary, then α still an isomorphism.

Proof. First we show that α is linear. Using the fact that $d(\pi_k)_p$ is linear (Proposition 3.6a), we get

$$\begin{aligned} \alpha(av + bw) &= (d(\pi_1)_p(av + bw), \ldots d(\pi_k)_p(av + bw)) \\ &= (ad(\pi_1)_p(v) + bd(\pi_1)_p(w), \ldots ad(\pi_k)_p(v) + bd(\pi_k)_p(w)) \\ &= a(d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)) + b(d(\pi_1)_p(w), \ldots, d(\pi_k)_p(w)) \\ &= a\alpha(v) + b\alpha(w) \end{aligned}$$

so α is linear. We will show that α is invertible by exhibiting an inverse. First define

$$\begin{aligned} &\iota_i : M_i \to (M_1 \times \ldots \times M_k) \\ &\iota_i(x) = (p_1, p_2, \ldots x, \ldots p_k) \end{aligned}$$

where x is in the ith index. Now define

$$\beta : T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k \to T_p(M_1 \times \ldots \times M_k)$$
$$\beta(v_1, \ldots v_k) = \sum_{i=1}^{k} d(\iota_i)_{p_i}(v_i)$$

Note that

$$\pi_j \circ \iota_i(p_i) = \pi_j(p) = p_j$$

So $\pi_j \circ \iota_i$ is the identity on M_i when $i = j$ and a constant map when $i \neq j$. By Proposition 3.6(b), we have

$$d(\pi_j)_p d(\iota_i)_{p_i} = d(\pi_j \circ \iota_i)_{p_i}$$

Let $(v_1, \ldots v_k) \in \bigoplus_{i=1}^k T_{p_i}M_i$. Then

$$\alpha \circ \beta(v_1, \ldots v_k) = \alpha \left(\sum_{i=1}^{k} d(\iota_i)_{p_i}(v_i) \right) = \sum_{i=1}^{k} \alpha(d(\iota_i)_{p_i}(v_i))$$

For each i,

$$\begin{aligned} \alpha(d(\iota_i)_{p_i}(v_i)) &= (d(\pi_1)_p d(\iota_i)_{p_i}(v_i), \ldots d(\pi_k)_p d(\iota_i)_{p_i}(v_i)) \\ &= (d(\pi_1 \circ \iota_i)_{p_i}(v_i), \ldots d(\pi_k \circ \iota_i)_{p_i}(v_i)) \end{aligned}$$

As noted previously, $\pi_j \circ \iota_{M_i}$ is either the identity map (when $i = j$) or a constant map. As shown in Exercise 1, the differential of a constant map is zero, and using Proposition 3.6(c) we get

$$\alpha(d(\iota_{M_i})_{p_i}(v_i)) = (0, \ldots, \mathrm{Id}_{T_{p_i}M_i}(v_i), \ldots, 0) = (0, \ldots, v_i, \ldots, 0)$$

Returning to the computation of $\alpha \circ \beta$, we get

$$\alpha \circ \beta(v_1, \ldots, v_k) = \sum_{i=1}^{k} (0, \ldots v_i, \ldots, 0) = (v_1, \ldots, v_k)$$

hence $\alpha \circ \beta$ is the identity on $\bigoplus_{i=1}^k T_{p_i}M_i$. Thus α has a right inverse, so it is surjective. Note that $T_p(M_1 \times \ldots \times M_k)$ has dimension equal to the sum of the dimensions of the M_i, which is also equal to the dimension of $\oplus_{i=1}^k T_{p_i}M_i$. Thus α is a surjective map between vector spaces of the same dimension, so it is a bijection. Thus α is an isomorphism. □

Proposition 0.7 (Exercise 3-6). *Consider S^3 as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z_1, z_2) \in S^3$, define a curve $\gamma_z : \mathbb{R} \to S^3$ by $\gamma_z(t) = (e^{it}z_1, e^{it}z_2)$. Then γ_z is a smooth curve whose velocity is never zero.*

Proof. (Throughout, we refrain from using i as an index, and only use it to refer to the imaginary unit.) Let $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ be the projections $(z_1, z_2) \mapsto z_1$ and $(z_1, z_2) \mapsto z_2$. Considering γ_z as a curve into $\mathbb{C}^2$, we note that the compositions

$$\begin{aligned} \pi_1 \circ \gamma_z &: \mathbb{R} \to \mathbb{C} \qquad & \pi_1 \circ \gamma_z(t) &= e^{it} z_1 \\ \pi_2 \circ \gamma_z &: \mathbb{R} \to \mathbb{C} \qquad & \pi_2 \circ \gamma_z(t) &= e^{it} z_2 \end{aligned}$$

are each smooth in the standard analytic sense, so by Proposition 2.12 (of Lee), γ_z is smooth. Now we compute the velocity of γ_z at t_0. First note that

$$\begin{aligned} \frac{d\gamma_z^1}{dt}(t_0) &= ie^{it} z_1\big|_{t=t_0} = ie^{it_0} z_1 \\ \frac{d\gamma_z^2}{dt}(t_0) &= ie^{it} z_2\big|_{t=t_0} = ie^{it_0} z_2 \end{aligned}$$

Then we compute the velocity as

$$\gamma_z'(t_0) = \frac{d\gamma_z^k}{dt}(t_0) \left.\frac{\partial}{\partial x^k}\right|_{\gamma_z(t_0)} = (ie^{it_0}) \left(z_k \left.\frac{\partial}{\partial x^k}\right|_{\gamma_z(t_0)} \right)$$

We know that $e^{it_0} \neq 0$ for any $t_0 \in \mathbb{R}$. Since $\frac{\partial}{\partial x^k}\big|_{\gamma_z(t_0)}$ is a basis for $T_{\gamma_z(t_0)} S^3$, we have

$$0 = z_k \left.\frac{\partial}{\partial x^k}\right|_{\gamma_z(t_0)} \iff \forall k, z_k = 0$$

Since $(z_1, z_2) \in S^3$, we can't have $z_1 = 0$ or $z_2 = 0$. Thus $\gamma_z'(t_0)$ is never zero. $\square$